TSFS02 — Vehicle Dynamics and Control

Computer Exercise 4: Stability and Control of a Car-Trailer System

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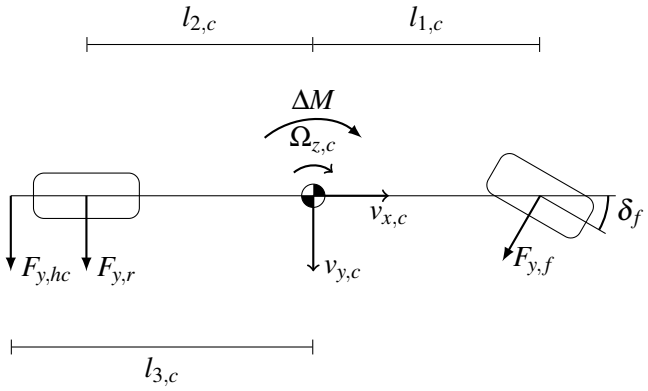

Figure 1 Single-track car model with hitch force $F_{y,hc}$ and torque vectoring moment ΔM.

1 Introduction

In this exercise you will familiarize yourself with a car-trailer system. The purpose of the exercise is to get an understanding of stability and control of vehicle dynamics systems.

1.1 Examination

To pass this exercise you should have fulfilled the following:

- Solved the preparation tasks.
- Solved all the computer exercise tasks.
- Answered all questions, with motivated and thoughtful answers.

The examination is done by presenting your results and answers to a course assistant at the scheduled exercise session. To speed up the examination process, it is recommended to present the tasks as you complete them, instead of saving them all to the end.

1.2 Prior Knowledge and Skills

To complete all tasks you need to:

- Be able to work with MATLAB .
- Understand and be able to sketch the single-track model with a trailer.
- Understand the linear tire model.

2 Preparation Tasks

Task 1 – Equations of motions for car

For the single-track car model in Figure 1, write down the equations of motion describing the lateral velocity $v_{y,c}$ and yaw rate $\Omega_{z,c}$. Include the lateral hitch force acting on the car $F_{y,hc}$ and the torque vectoring moment ΔM (active moment contribution as in Lab 3).

Task 2 – Equations of motions for trailer

For the single-track trailer model in Figure 2, write down the equations of motion describing the lateral velocity $v_{y,t}$ and yaw rate $\Omega_{z,t}$.

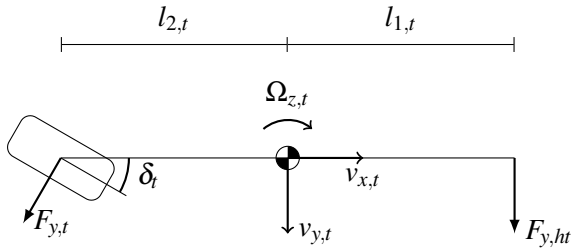

Figure 2 Single-track trailer model with trailer steering.

Task 3 – Kinematics

You will now consider the kinematics of the car-trailer system. The car and trailer are attached at the hitch joint with an angle Γ.

a) Find an expression for $\Omega_{z,t}$ in terms of the the hitch angle rate $\dot{\Gamma}$ the car yaw rate $\Omega_{z,c}$.

b) Find a linear expression for $\dot{v}_{y,t}$ in terms of $\dot{v}_{y,c}$, $\dot{\Omega}_{z,c}$, and $\ddot{\Gamma}$. To keep the model linear you should in this exercise assume $\Gamma = 0$.

Hint 1: First calculate the lateral velocity at the hitch joint.

Hint 2: Let $(\dot{v}_{y,t})_{\text{trailer}}$ denote the time derivative in the reference frame of the trailer, then in the reference frame of the car the time derivative is $(\dot{v}_{y,t})_{\text{car}} = (\dot{v}_{y,t})_{\text{trailer}} - v_x\dot{\Gamma}$.

c) Find linear expressions for the slip angles for each tire in terms of δ_f, δ_t, Γ, $v_{y,c}$, $\Omega_{z,c}$ and $\dot{\Gamma}$. Assume $v_{x,t} = v_{x,c} = v_x$ is a parameter and use $\arctan x \approx x$ for small x.

Task 4 – Car-trailer system

You will now calculate the linearized equations of motion for the car-trailer system under the following assumptions:

I Assume the longitudinal velocities are constant and equal $v_{x,c} = v_{x,t} = v_x$.

II Assume small steering angles such that $\cos(\delta_i) = 1$ and $\sin(\delta_i) = \delta_i$.

III Assume the hitch forces are related by $F_{y,hc} = -F_{y,ht}$.

IV Use the linear tire model $F_{y,i} = 2k_i\alpha_i$ where k_i is the tire stiffness.

For the state vector $x = [x_1, x_2, x_3, x_4]^T = [v_{y,c}, \Omega_{z,c}, \dot{\Gamma}, \Gamma]^T$, write the linear system of equations on the form:

$$M\dot{x} = D(v_x)x + C_{\delta_f}\delta_f + C_{\delta_t}\delta_t + C_{\Delta M}\Delta M. \tag{1}$$

To simplify, the matrices M, $D(v_x)$, C_{δ_f}, C_{δ_t} and $C_{\Delta M}$ will automatically be created during the lab by MATLAB 's symbolic toolbox provided you supply the system as four linear equations $0 = f_i(\dot{x}, x)$. You may divide the calculations into smaller components, *e.g.*, provide a function f_i as a function of the tire forces, provided that you first write down an expression of the tire forces as a function of x and $\dot{x}$.

Hints: Use one of the equations from Task 1 or Task 2 to eliminate the hitch forces $F_{y,hc}$ and $F_{y,ht}$ from the equations of motion for the car-trailer system. One of the equations f_i is $f_4 = -\dot{x}_4 + x_3$.

3 Exercise Tasks

In the following tasks you will examine the linear model you derived in the preparatory tasks and simulate a comparable nonlinear model.

Start by downloading the prepared MATLAB files from the homepage (TSFS02_Lab4_CarTrailer.zip).

Task 5 – Implement linear system

Open the file `getLinearSystem.m` and enter your equations for $f_i(\dot{x},x)$ from Task 4. Run the script `testLinearSystem.m`, ensure the printed variables `dx` and `dx_correct` are the same, if not, check your equations again.

Task 6 – Stability of uncontrolled system

You will now analyze the stability of the car-trailer system. To do this you will want to rewrite your system (1) to the form

$$\dot{x} = A(v_x)x + B_{\delta_f}\delta_f + B_{\delta_t}\delta_t + B_{\Delta M}\Delta M. \tag{2}$$

The uncontrolled system is then $\dot{x} = A(v_x)x$. In MATLAB you can efficiently compute $M^{-1}D$ with `M\D`.

The damping ratio ζ is a good indicator of dynamic stability. It is given by

$$\zeta = \frac{-d}{\sqrt{d^2+\omega_d^2}}, \tag{3}$$

where d and ω_d are the real and imaginary parts of the system eigenvalues, respectively.

To compute the eigenvalues use the function `eigs`. The symbolic expression needs to be replaced with numerical values. To replace symbolic values with numerical ones use the MATLAB function `subs` and use the function `double` to change the type of the variable, *e.g.*, `A = double(subs(Asym,p))`. See the documentation in MATLAB for more information.

a) Open the file `dampingRatio.m` and plot the damping ratios of the uncontrolled system as a function of speed. At what velocity does the system become unstable? Open the file `simulation.m` and investigate the dynamic behavior around the unstable velocity.

b) From the lecture you may remember that the stability of a trailer can be increased by moving the center of percussion forward in the trailer. Examine what happens to the damping ratios if you move the center of mass of the trailer (keep $L_t = l_{1,t} + l_{2,t}$ the same). At what velocity does the system become unstable for $l_{2,t} = 0.5$? How does the instability differ from the one in a)? Compare them by simulation.

c) Keep $l_{2,t} = 0.5$. Modify the mass and moment of inertia of the trailer by the same factor. How does this effect the stability of the car-trailer system? Use the results to try to explain why the instability from b) occurs.

Task 7 – Stabilization through active control

The dynamics of a linear actively controlled closed-loop system with input $u = -Lx$ can be written as

$$\dot{x} = (A - BL)x. \tag{4}$$

In this task you will use the linear-quadratic regulator (LQR) technique to design controllers for the vehicle-trailer system. The LQR technique finds the L in (4) that minimizes the quadratic optimization criterion

$$J = \int_0^\infty (x^T Q x + u^T R u)\,\mathrm{d}t. \tag{5}$$

Design two LQR controllers, one that steers the trailer wheel and one that uses active torque distribution. To your help, use the MATLAB function `L = lqr(A,B,Q,R)`. Let $R = 1$ and Q to be a diagonal matrix.

Implement your LQR controllers both in `dampingRatio.m` and `simulation.m` and examine their behavior. How do the controllers perform at different vehicle speeds? Which states are best controlled by each controller?

Hint: Make large changes to Q when tuning your controllers.

A Nomenclature

Table 1 lists all the parameters that are available in `params.mat`.

Table 1 Parameters.

Parameter	Description	Value	Unit
`mc`	Car mass	1700	kg
`Ic`	Car yaw inertia	2000	kgm^2
`l1c`	Distance from car CoM to front axle	1.5	m
`l2c`	Distance from car CoM to rear axle	1.7	m
`l3c`	Distance from car CoM to hitch	2.9	m
`mt`	Trailer mass	2000	kg
`It`	Trailer yaw inertia	3000	kgm^2
`l1t`	Distance from trailer CoM to hitch	6	m
`l2t`	Distance from trailer CoM to trailer axle	0	m
`kf`	Front wheel tire stiffness	40 000	N/rad
`kr`	Rear wheel tire stiffness	40 000	N/rad
`kt`	Trailer wheel tire stiffness	40 000	N/rad